

To the question of a nonrelativistic wave equation for a system of interacting particles

M. V. Kuzmenko

*Bogolyubov Institute for Theoretical Physics of the NAS of Ukraine,
Metrolohichna Str., 14b, Kyiv-143, 252143 Ukraine*

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Abstract

It is shown that the Schrödinger nonrelativistic equation of a system of interacting particles is not a rigorously nonrelativistic equation since it is based on the implicit assumption of finiteness of the interaction propagation velocity. For a system of interacting particles, a fully nonrelativistic nonlinear system of integro-differential equations is proposed. In the case where the size of the system of particles is of the same order as the Compton wavelength associated with particles, certain essential differences are shown to exist as compared with traditional consequences of the nonrelativistic Schrödinger equation.

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I. INTRODUCTION

In classical mechanics, the problem of two particles, which interact with force depending only on the relative distance between them, is separated into two three-dimensional problems: the free-particle problem and one of a particle in a static potential field [1]. Motion of a free particle, whose mass is equal to the sum of the masses of both particles, is a motion of the center of masses of the system and is uniform and rectilinear. Motion of a relative particle with so-called reduced mass occurs in the field with potential $V(\mathbf{r})$.

An analogous situation is also observed for the nonrelativistic wave equation of a system of two interacting particles, which was proposed by Schrödinger already in the second communication on wave mechanics by the example of elastic rotator (two-atom molecule) [2].

The probabilistic interpretation of the square of the modulus of a wave function is possible only under the assumption that measurements of coordinates or momenta of various particles do not principally disturb one another even if there exists some interaction between particles [3]. This means that operators of coordinates and momenta of two particles commute with each other. But in the theory of Schrödinger, still operators of coordinates and momenta of various particles commute with one another that is equivalent to the lacking of any interference on measurement of the coordinate of one particle and the momentum of the other. The last assertion is valid if the duration of measurement of the coordinate of a particle is much less than the duration of propagation of a light signal at distances of about the size of the system, or, what is the same, if the Compton wave length is much less than the size of the system. Therefore, the Schrödinger equation perfectly works in atomic physics and solid-state physics. But a direct application of the Schrödinger equation to atomic nuclei seems not entirely correct because the Compton wave length of a nucleon is of order of the size of an atomic nucleus itself. In addition, a rigorous nonrelativistic statement requires to consider the interaction propagation velocity to be infinitely large that forces us to assume that operators of coordinates and momenta of various particles do not commute with one another.

II. A COMPLETELY NONRELATIVISTIC STATEMENT OF THE QUANTUM TWO-BODY PROBLEM

As is known, classical equations of motion of a particle with mass m in an external field $V(\mathbf{r})$ follow from the Hamilton function

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}), \quad (1)$$

which depends on the coordinates \mathbf{r} of the particle and on the corresponding momentum \mathbf{p} . The total energy of the system

$$E = H(\mathbf{r}, \mathbf{p}). \quad (2)$$

With this classical system, we associate a quantum system, whose dynamic state is represented by the wave function $\Psi(\mathbf{r}, t)$ defined in the configuration space. The wave equation is deduced by the formal substitution of the quantities E , \mathbf{r} , \mathbf{p} in both sides of relation (2) by the corresponding operators [4]:

$$E \rightarrow \hat{E} = i\hbar \frac{\partial}{\partial t}, \quad (3)$$

$$\mathbf{r} \rightarrow \hat{\mathbf{r}} = \mathbf{r}, \quad (4)$$

$$\mathbf{p} \rightarrow \hat{\mathbf{p}} = -i\hbar \nabla_{\mathbf{r}}. \quad (5)$$

Here, $\hbar = \frac{h}{2\pi}$, where h is the universal constant introduced by Planck.

It is meant that the result of action of both sides of equality (2), considered as operators, on $\Psi(\mathbf{r}, t)$ is the same. The realization of this fact implies the Schrödinger nonrelativistic equation for a particle in an external field $V(\mathbf{r})$:

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left[-\frac{\hbar^2}{2m} \Delta + V(\mathbf{r}) \right] \Psi(\mathbf{r}, t). \quad (6)$$

It is worth to emphasize that the operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ in (4), (5) are written in the configuration space, and \mathbf{r} is the vector of position of the particle in the Cartesian coordinate system.

Operators of coordinate and momentum do not commute with each other:

$$[\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{z}, \hat{p}_z] = i\hbar, \quad (7)$$

that leads to the Heisenberg uncertainty relations

$$\Delta x \Delta p_x \geq \hbar/2, \quad \Delta y \Delta p_y \geq \hbar/2, \quad \Delta z \Delta p_z \geq \hbar/2, \quad (8)$$

where the quantities Δx , Δp_x , Δy , Δp_y , Δz , and Δp_z are directly connected with corresponding measurements and present mean square deviations from the mean value. For example, we have

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} \quad (9)$$

for the coordinate x , by definition, where $\langle \hat{A} \rangle$ is the mean value of the operator \hat{A} in the dynamic state defined by a wave function $\Psi(\mathbf{r}, t)$.

Relations (8) assert that a particle cannot be in states, in which its coordinate and momentum simultaneously take quite definite, exact values. Moreover, quantum theory accepts that an unpredictable and uncontrolled perturbation, undergone by a physical system in the process of measurement, is always finite and such that the Heisenberg uncertainty relations (8) are satisfied [4]. Hence, no experiment can lead to a simultaneous exact measurement of the coordinate and momentum of a particle. For example, the measurement of the coordinate x with accuracy Δx in the well-known experiment with a microscope, considered by Heisenberg, is accompanied by the uncontrolled transfer of momentum to the particle, which is characterized by the uncertainty

$$\Delta p_x \sim \frac{\hbar}{2\Delta x}. \quad (10)$$

In this case, limits of the accuracy of determination of a position are set always by the optical resolution stipulated by diffraction effects according to classical wave optics. For

example, it is known that the limit accuracy of an image Δx for a microscope is defined by the formula

$$\Delta x \sim \frac{\lambda'}{\sin \vartheta}, \quad (11)$$

where λ' is the wave length of scattered light, which can differ from that of incident light, and ϑ is half the objective aperture. According to relation (11), to increase the accuracy, it is profitable to have the wave length of scattered light as short as possible. But owing to the Compton effect, the frequency of scattered light changes by a value defined by the conservation laws of energy and momentum. This implies that, even in the limit $\nu \rightarrow \infty$ ($\lambda = c/\nu \rightarrow 0$), the frequency of scattered emission ν' cannot exceed some finite value. If \mathbf{p} is a momentum, \mathbf{v} is a velocity, and $E = c\sqrt{m^2c^2 + p^2}$ is the energy of a material particle prior to the process of scattering, then we have [3]

$$\nu' = \frac{mc^2}{h} \frac{1}{\sqrt{1 - v^2/c^2}}, \quad (12)$$

$$\lambda' = \frac{h}{mc} \sqrt{1 - v^2/c^2} \quad (13)$$

in this limit, which gives a maximum value for ν' and a minimum one for $\lambda' = c/\nu'$. Here, c is the velocity of light in vacuum.

Thus, to attain the maximum accuracy of determination of a position on the observation of the scattering of a quantum of light by using an optical instrument, we obtain the following expression:

$$\Delta x = \frac{h}{mc} \sqrt{1 - v^2/c^2}. \quad (14)$$

The duration of the process of measurement of a position, i.e., the time, during which the interaction between a light quantum and a particle can occur, can in no way be less than the periods of oscillations of the incident and scattered emissions and should be of the order of $1/\nu'$:

$$\Delta t = \frac{h}{mc^2} \sqrt{1 - v^2/c^2}. \quad (15)$$

If the size of the system is such that a characteristic time of flight for the system significantly exceeds Δt , then one can say that the process of measurement of the particle coordinate with accuracy Δx is followed by an impact on a particle with the force

$$F_x \sim \frac{\Delta p_x}{\Delta t} \sim \frac{\hbar c}{2(\Delta x)^2}. \quad (16)$$

Here, we assume that the momentum transferred to a particle under measurement of its coordinate is of order of the mean square deviation Δp_x .

On the measurement of the momentum of a particle with accuracy Δp_x , it undergoes an impact with force

$$F_x \sim \frac{2c}{\hbar} (\Delta p_x)^2. \quad (17)$$

Consider now a system of two interacting particles, whose Hamilton function is

$$H = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(|\mathbf{r}_2 - \mathbf{r}_1|), \quad (18)$$

where \mathbf{r}_1 and \mathbf{r}_2 are Cartesian coordinates of two particles with masses m_1 and m_2 , \mathbf{p}_1 and \mathbf{p}_2 are their corresponding momenta, and the potential energy depends only on the distance between particles. To derive the nonrelativistic Schrödinger equation for this system, we proceed analogously to Eq. (6).

This classical system is put into correspondence to the quantum system, whose dynamic state is represented by the wave function $\Psi(\mathbf{r}_1, \mathbf{r}_2, t)$ defined in the configuration space. The wave equation can be derived by the formal replacement of the quantities E , \mathbf{r}_1 , \mathbf{r}_2 , \mathbf{p}_1 , and \mathbf{p}_2 on both sides of a relation analogous to (2) by the relevant operators

$$E \rightarrow \hat{E} = i\hbar \frac{\partial}{\partial t}, \quad (19)$$

$$\mathbf{r}_1 \rightarrow \hat{\mathbf{r}}_1 = \mathbf{r}_1, \quad (20)$$

$$\mathbf{r}_2 \rightarrow \hat{\mathbf{r}}_2 = \mathbf{r}_2, \quad (21)$$

$$\mathbf{p}_1 \rightarrow \hat{\mathbf{p}}_1 = -i\hbar \nabla_1, \quad (22)$$

$$\mathbf{p}_2 \rightarrow \hat{\mathbf{p}}_2 = -i\hbar \nabla_2. \quad (23)$$

Then the well-known Schrödinger nonrelativistic equation for a system of two interacting particles has the form

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \left[-\frac{\hbar^2}{2m_1} \Delta_1 - \frac{\hbar^2}{2m_2} \Delta_2 + V(|\mathbf{r}_2 - \mathbf{r}_1|) \right] \Psi(\mathbf{r}_1, \mathbf{r}_2, t). \quad (24)$$

The operators $\hat{\mathbf{r}}_1$, $\hat{\mathbf{r}}_2$, $\hat{\mathbf{p}}_1$, and $\hat{\mathbf{p}}_2$ are such that they satisfy the following commutation relations:

$$[\hat{x}_k, \hat{p}_{kx}] = i\hbar, \quad [\hat{y}_k, \hat{p}_{ky}] = i\hbar, \quad [\hat{z}_k, \hat{p}_{kz}] = i\hbar, \quad k = 1, 2. \quad (25)$$

All other possible commutation relations equal zero, including

$$[\hat{x}_k, \hat{p}_{lx}] = 0, \quad [\hat{y}_k, \hat{p}_{ly}] = 0, \quad [\hat{z}_k, \hat{p}_{lz}] = 0, \quad k, l = 1, 2, \quad k \neq l. \quad (26)$$

Equalities (26) are based on the assumption that measurements of coordinates and momenta of different particles do not disturb one another in principle even in the presence of some interaction forces between particles [3]. That is, one supposes that the change in the force action of a particle on another one, caused by a measurement of the coordinate of the first, propagates with finite velocity.

Thus, to derive the Schrödinger nonrelativistic equation for a two-particle system, one uses, on the one hand, the Hamilton classical nonrelativistic function and, on the other hand, the implicit assumption about finiteness of the interaction propagation velocity.

In the fully nonrelativistic quantum theory, we must consider the interaction propagation velocity as infinitely large, which forces us to drop the requirement for the commutation

relations (26) to hold. Having accepted this viewpoint, we will consider that, under a measurement of the coordinate of the first particle, there occurs the uncontrolled transfer of momentum not only to this particle but to the whole system since the particles are connected through the interaction potential, whose propagation velocity is infinitely large. Therefore, it is natural to demand that the commutator of the operator of the coordinate of the first particle with the operator of the total momentum of the system be equal to $i\hbar$:

$$[\hat{x}_1, \hat{P}_{cx}] = i\hbar, \quad [\hat{y}_1, \hat{P}_{cy}] = i\hbar, \quad [\hat{z}_1, \hat{P}_{cz}] = i\hbar. \quad (27)$$

Here, $\hat{\mathbf{P}}_c = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2$ is the operator of the total momentum of the system. The same should be true for the second particle:

$$[\hat{x}_2, \hat{P}_{cx}] = i\hbar, \quad [\hat{y}_2, \hat{P}_{cy}] = i\hbar, \quad [\hat{z}_2, \hat{P}_{cz}] = i\hbar. \quad (28)$$

Note that relations (27) and (28) hold true also for a Schrödinger nonrelativistic equation. Namely, they allow one to construct the operator of coordinates of the center of masses of the system. The commutator of the last with the operator of the total momentum of the system equals $i\hbar$. On the contrary, the fulfillment of relations (25) is not obligatory for a system of interacting particles, and we intend to reject this requirement.

It is clear that, on measuring the coordinate of some particle with accuracy Δx , the system undergoes an impact with force $\sim \hbar c / 2(\Delta x)^2$. For example, the measurement of the coordinate of a nonrelativistic electron under observation of the scattering of a light quantum with an optical device with the greatest possible accuracy of order of the Compton wave length $\lambda_e = h/m_e c = 2.4 \cdot 10^{-10}$ cm is accompanied by impact with the force $F_e \sim 10^8$ MeV/cm. For a proton with its Compton wave length of the order of $1.3 \cdot 10^{-13}$ cm, the impact force is about $F_p \sim 10^{15}$ MeV/cm. The mean interaction force between particles in a hydrogen atom in the ground state is $F_H \sim 10^4$ MeV/cm, and that for the bound state of the nucleus of deuterium is $F_D \sim 10^{14}$ MeV/cm. Therefore, whereas we can neglect the interaction force $F_H/F_e \sim 10^{-4}$ between particles on measuring the coordinates of particles in an atom and consider the operators of coordinates and momenta of various particles to be commuting, it is not the case for an atomic nucleus, because the ratio of the interparticle interaction force to the impact one is of order $F_D/F_p \sim 10^{-1}$.

In the general case, let

$$[\hat{x}_1, \hat{p}_{2x}] = i\hbar \hat{f}_1, \quad (29)$$

where \hat{f}_1 is some dimensionless Hermitian operator. Then it follows from Eq. (27) that

$$[\hat{x}_1, \hat{p}_{1x}] = i\hbar(1 - \hat{f}_1). \quad (30)$$

By analogy, if

$$[\hat{x}_2, \hat{p}_{1x}] = i\hbar \hat{f}_2, \quad (31)$$

then

$$[\hat{x}_2, \hat{p}_{2x}] = i\hbar(1 - \hat{f}_2). \quad (32)$$

The dimensionless Hermitian operators \hat{f}_1 and \hat{f}_2 depend generally on the interaction force between particles \mathbf{F}_{12} and on masses of the interacting particles m_1 and m_2 . The operators \hat{f}_1 and \hat{f}_2 cannot depend on a direction of the vector \mathbf{F}_{12} , since the commutation relations for the x , y , and z components should be identical by analogy with (29)-(32), since there are no separated directions in the system, and independent variables in the Cartesian coordinate system are fully equivalent. For this reason, the operators \hat{f}_1 and \hat{f}_2 are only functions of the absolute value of a force, i.e., of F_{12}^2 :

$$\hat{f}_1 \equiv \hat{f}_1(m_1, m_2, F_{12}^2), \quad \hat{f}_2 \equiv \hat{f}_2(m_1, m_2, F_{12}^2). \quad (33)$$

Let us make permutation of m_1 and m_2 . Then

$$[\hat{x}_1, \hat{p}_{2x}] = i\hbar \hat{f}_1(m_2, m_1, F_{12}^2), \quad [\hat{x}_2, \hat{p}_{1x}] = i\hbar \hat{f}_2(m_2, m_1, F_{12}^2). \quad (34)$$

Compare (34) with (29), (31). Considering that the physical situation has not changed, we get

$$\hat{f}_1(m_2, m_1, F_{12}^2) = \hat{f}_2(m_1, m_2, F_{12}^2). \quad (35)$$

Thus, we have one unknown operator $\hat{f}_1(m_1, m_2, F_{12}^2)$. For $m_2 \rightarrow 0$, \hat{f}_1 must tend to zero since, in the absence of the second particle, the whole momentum transferred under the measurement of the coordinate x_1 falls namely this particle. If $F_{12} \rightarrow 0$, then $\hat{f}_1 \rightarrow 0$, i.e., without any interaction forces between particles, the operators of coordinates and momenta of different particles commute among themselves. The situation $F_{12} \rightarrow \infty$ corresponds to the case where we have one particle of mass M and mentally represent that it consists of two strongly bound particles with masses m_1 and m_2 . Therefore, the momentum, received under a measurement of some coordinate, is distributed proportionally to masses of particles. This enables us to write down \hat{f}_1 as $\hat{f}_1 = m_2/M$. Here, $M = m_1 + m_2$ is the system mass.

Therefore, without loss of generality, we can present the operator \hat{f}_1 as

$$\hat{f}_1 = \frac{m_2}{M} \hat{\varepsilon}(F_{12}^2, m_1, m_2), \quad (36)$$

where $\hat{\varepsilon}$ is a new operator, which is assumed to be symmetric with respect to the masses of particles m_1 and m_2 . In what follows, we will omit its explicit dependence on masses to shorten formulas, namely, $\hat{\varepsilon}(F_{12}^2, m_1, m_2) \equiv \hat{\varepsilon}(F_{12}^2)$. For $F_{12} \rightarrow 0$, $\hat{\varepsilon} \rightarrow 0$, and $\hat{\varepsilon} \rightarrow 1$ for $F_{12} \rightarrow \infty$.

For the noncommuting operators \hat{x}_1 and \hat{p}_{2x} , the uncertainty relation has the form [5]

$$\Delta x_1 \Delta p_{2x} \geq \frac{\hbar m_2}{2 M} \left| \langle \hat{\varepsilon}(F_{12}^2) \rangle \right|, \quad (37)$$

where $\langle \hat{\varepsilon}(F_{12}^2) \rangle \equiv \varepsilon$ is a quantum-mechanical average in the state $\Psi(\mathbf{r}_1, \mathbf{r}_2, t)$.

If we replace the operator $\hat{\varepsilon}$ in (36) by its averaged quantum-mechanical value $\langle \hat{\varepsilon}(F_{12}^2) \rangle$, the uncertainty relation (37) does not change. This makes it possible to write down a non-relativistic wave equation for a two-particle system, since the operator \hat{f}_1 is now a constant.

It is worth to say several words about the commutativity of the operators of coordinates and momenta of different particles between themselves:

$$[\hat{x}_1, \hat{x}_2] = 0, \quad [\hat{p}_{1x}, \hat{p}_{2x}] = 0. \quad (38)$$

If we increase the accuracy of measurements of the coordinates x_1, x_2 , the impact forces $F_1 = \hbar c/2(\Delta x_1)^2$ and $F_2 = \hbar c/2(\Delta x_2)^2$ also grow. For $\Delta x_1 \rightarrow 0$ and $\Delta x_2 \rightarrow 0$, we have $F_1 \gg F_{12}$, $F_2 \gg F_{12}$. Therefore, we can neglect the interaction force F_{12} between particles and consider the operators of coordinates of both particles as commuting. Analogously, on measuring the momenta p_{1x}, p_{2x} , the impact forces $F_1 = 2c(\Delta p_{1x})^2/\hbar$ and $F_2 = 2c(\Delta p_{2x})^2/\hbar$ tend to zero on increasing the accuracy of measurements. For this reason, the operators \hat{p}_{1x} and \hat{p}_{2x} also can be considered as commuting.

We present now the commutation relations for all operators of coordinates and momenta in the two-body problem:

$$[\hat{x}_1, \hat{p}_{1x}] = i\hbar \left(1 - \frac{m_2}{M}\varepsilon\right), \quad (39)$$

$$[\hat{x}_2, \hat{p}_{2x}] = i\hbar \left(1 - \frac{m_1}{M}\varepsilon\right), \quad (40)$$

$$[\hat{x}_1, \hat{p}_{2x}] = i\hbar \frac{m_2}{M}\varepsilon, \quad (41)$$

$$[\hat{x}_2, \hat{p}_{1x}] = i\hbar \frac{m_1}{M}\varepsilon, \quad (42)$$

$$[\hat{x}_1, \hat{x}_2] = 0, \quad (43)$$

$$[\hat{p}_{1x}, \hat{p}_{2x}] = 0. \quad (44)$$

For the y and z components, we have analogous relations. We recall that ε is a quantum mechanical mean value of the operator $\hat{\varepsilon}(F_{12}^2)$ in the state $\Psi(\mathbf{r}_1, \mathbf{r}_2, t)$:

$$\varepsilon = \frac{\langle \Psi, \hat{\varepsilon}(F_{12}^2) \Psi \rangle}{\langle \Psi, \Psi \rangle}. \quad (45)$$

We can construct now one of the possible representations for the operators of coordinates and momenta of a two-particle system:

$$\hat{\mathbf{r}}_1 = -(1 - \varepsilon) \frac{m_2}{M} \mathbf{r} + \mathbf{R}, \quad (46)$$

$$\hat{\mathbf{r}}_2 = (1 - \varepsilon) \frac{m_1}{M} \mathbf{r} + \mathbf{R}, \quad (47)$$

$$\hat{\mathbf{p}}_1 = i\hbar \nabla_{\mathbf{r}} - i\hbar \frac{m_1}{M} \nabla_{\mathbf{R}}, \quad (48)$$

$$\hat{\mathbf{p}}_2 = -i\hbar \nabla_{\mathbf{r}} - i\hbar \frac{m_2}{M} \nabla_{\mathbf{R}}. \quad (49)$$

It is easily to verify that operators (46)-(49) satisfy the commutation relations (39)-(44).

In (46)-(49), \mathbf{r} and \mathbf{R} are independent operator variables. The latter represents coordinates of the center of masses of the system:

$$\hat{\mathbf{r}}_c = \frac{m_1 \hat{\mathbf{r}}_1 + m_2 \hat{\mathbf{r}}_2}{m_1 + m_2} = \mathbf{R}. \quad (50)$$

The operator of the total momentum of the system is

$$\hat{\mathbf{P}}_c = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2 = -i\hbar\nabla_{\mathbf{R}}. \quad (51)$$

By substituting operators (46)-(49) to the Hamilton function (18), we get the nonrelativistic wave equation for a two-particle system:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}, \mathbf{R}, t) = \left[-\frac{\hbar^2}{2M}\Delta_{\mathbf{R}} - \frac{\hbar^2}{2\mu}\Delta_{\mathbf{r}} + V[r(1-\varepsilon)] \right] \Psi(\mathbf{r}, \mathbf{R}, t). \quad (52)$$

In this case, ε is defined according to (45), and μ is the reduced mass of the system, $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

For the Hamiltonian H not depending explicitly on time, the substitution

$$\Psi = \psi \exp\left(-i\frac{Et}{\hbar}\right), \quad (53)$$

where ψ depends on coordinates in the configuration space but not on time, implies the nonlinear system of integro-differential equations for stationary states of the two-particle system:

$$\left[-\frac{\hbar^2}{2M}\Delta_{\mathbf{R}} - \frac{\hbar^2}{2\mu}\Delta_{\mathbf{r}} + V[r(1-\varepsilon)] \right] \psi(\mathbf{r}, \mathbf{R}) = E\psi(\mathbf{r}, \mathbf{R}), \quad (54)$$

$$\varepsilon = \frac{\langle \psi, \hat{\varepsilon}(F_{12}^2) \psi \rangle}{\langle \psi, \psi \rangle}. \quad (55)$$

By the substitution $\psi(\mathbf{r}, \mathbf{R}) = \Phi(\mathbf{R}) \varphi(\mathbf{r})$, we can separate motions of the center of masses of the system as a whole. As a result, we obtained the following nonlinear system of equations:

$$\left[-\frac{\hbar^2}{2\mu}\Delta_{\mathbf{r}} + V[r(1-\varepsilon)] \right] \varphi(\mathbf{r}) = E\varphi(\mathbf{r}), \quad (56)$$

$$\varepsilon = \frac{\langle \varphi, \hat{\varepsilon}(F_{12}^2) \varphi \rangle}{\langle \varphi, \varphi \rangle}. \quad (57)$$

As in the Schrödinger nonrelativistic theory, a wave function $\varphi(\mathbf{r})$ should be continuous together with its partial derivatives of the first order in the whole space and, in addition, be a bounded single-valued function of its arguments.

As in the Schrödinger theory, for particles interacting by means of a centrally symmetric potential, which depends only on the distance between particles, the wave function $\varphi(\mathbf{r})$ can be represented in the following form:

$$\varphi(\mathbf{r}) = \frac{1}{r} \chi_l(r) Y_{lm}\left(\frac{\mathbf{r}}{r}\right), \quad (58)$$

where $Y_{lm}\left(\frac{\mathbf{r}}{r}\right)$ are orthonormalized spherical functions. Then the function $\chi_l(r)$ satisfies the following system of equations:

$$\left[-\frac{\hbar^2}{2\mu} \left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) + V[r(1-\varepsilon)] \right] \chi_l(r) = E \chi_l(r), \quad (59)$$

$$\varepsilon = \frac{\langle \chi_l, \hat{\varepsilon}(F_{12}^2) \chi_l \rangle}{\langle \chi_l, \chi_l \rangle}. \quad (60)$$

The quantity $\frac{m_2}{M} \varepsilon$ presents the share of the momentum transferred to the second particle on measuring the coordinate of the first. To construct the operator $\hat{\varepsilon}(F_{12}^2)$ on the basis of classical mechanics is an extremely difficult problem since one must be able to solve three-particle problems in the general form. Therefore, by taking into account the properties of the operator $\hat{\varepsilon}(F_{12}^2)$ at $F_{12} \rightarrow 0$ ($\hat{\varepsilon} \rightarrow 0$) and at $F_{12} \rightarrow \infty$ ($\hat{\varepsilon} \rightarrow 1$), it is convenient to approximate $1 - \hat{\varepsilon}$ in the first approximation by a gaussoid:

$$\hat{\varepsilon} = 1 - \exp \left(-\Omega_0 F_{12}^2 (|\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1|) \right), \quad (61)$$

where Ω_0 is a parameter with dimensionality inversely proportional to the square of force. The explanation for the definition of this will be considered below.

The nonlinear system of the integro-differential equations (59) - (61) for a two-particle system has solutions for definite values of the energy E , but solutions with different energies are not orthogonal to one another.

Moreover, since the parameter ε is a quantum-mechanical mean in every quantum state, we may say that every quantum state has its own potential of interaction between particles.

The constant Ω_0 can be defined by analyzing the discrete spectrum of a hydrogenlike atom. Let two particles with masses m_1 (electron) and m_2 (atomic nucleus) be coupled through the Coulomb potential $V(r) = -\frac{Ze^2}{r}$, where Z is the charge of the atomic nucleus. The nonlinear system of integro-differential equations for bound states (59) - (61) can be written in the following way:

$$\left[-\frac{\hbar^2}{2\mu} \left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) - \frac{Ze^2}{r(1-\varepsilon_{nl})} \right] \chi_{nl}(r) = E_{nl} \chi_{nl}(r), \quad (62)$$

$$\varepsilon_{nl} = 1 - \int_0^\infty \chi_{nl}^2(r) \exp \left(-\Omega_0 \frac{Z^2 e^4}{r^4 (1-\varepsilon_{nl})^4} \right) dr, \quad (63)$$

$$\int_0^\infty \chi_{nl}^2(r) dr = 1. \quad (64)$$

Here, μ is the reduced mass of the system.

Equations (62) and (64) define normalized radial functions of a hydrogenlike atom by the Schrödinger theory. Their solutions for bound states are well known (see, e.g., [6]):

$$\chi_{nl}(r) = N_{nl} r^{l+1} F \left(-n+l+1, 2l+2, \frac{2Zr}{(1-\varepsilon_{nl})na_0} \right) \exp \left(\frac{Zr}{(1-\varepsilon_{nl})na_0} \right), \quad (65)$$

where

$$N_{nl} = \frac{1}{(2l+1)!} \left[\frac{(n+l)!}{2n(n-l-1)!} \right]^{1/2} \left(\frac{2Z}{(1-\varepsilon_{nl})na_0} \right)^{l+3/2}. \quad (66)$$

Here, $a_0 = \frac{\hbar^2}{\mu e^2}$ is the Bohr radius, and F is a degenerate hypergeometric function.

Eigenvalues of energy are expressed as

$$E_{nl} = -\frac{\mu c^2 (\alpha Z)^2}{2} \frac{1}{n^2 (1 - \varepsilon_{nl})^2}. \quad (67)$$

Here, $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant, $l = 0, 1, \dots, n-1$, and $n = 1, 2, \dots, \infty$.

By substituting $\chi_{nl}(r)$ into Eq. (63), we obtain the nonlinear equation for the determination of ε_{nl} :

$$\eta_{nl} = S_{nl} \int_0^\infty x^{2l+2} \exp\left(-x - \frac{\Omega(\alpha Z)^6}{\eta_{nl}^8 n^4 x^4}\right) F^2(-n+l+1, 2l+2, x) dx, \quad (68)$$

where $\eta_{nl} = 1 - \varepsilon_{nl}$, $S_{nl} = \frac{1}{[(2l+1)!]^2} \left[\frac{(n+l)!}{2n(n-l-1)!} \right]$ and $\Omega = \frac{16\Omega_0(\mu c^2)^4}{\hbar^2 c^2}$.

For the ground state of a hydrogenlike atom, we have

$$\eta_{10} = \frac{1}{2} \int_0^\infty x^2 \exp\left(-x - \frac{\Omega(\alpha Z)^6}{\eta_{10}^8 x^4}\right) dx. \quad (69)$$

For η_{10} , the nonlinear Eq. (69) has solutions if $\Omega(\alpha Z)^6 \leq \Omega_c = 0.40765$ that is depicted in Fig. 1. From two solutions, that solution is considered as suitable which is located nearer to 1. The second solution should be omitted. Indeed, it corresponds to the case where ε tends to 1 as the parameter αZ decreases to zero, which contradicts to the assumptions made above about the parameter of noncommutativity. For $\Omega(\alpha Z)^6 > \Omega_c$, Eq. (69) has no solutions, which means it is impossible for a given bound state to exist.

The system of Eqs. (62)-(64) describes a nonrelativistic motion of a particle with mass μ in the external field $V(r) = -\frac{Ze^2}{r}$ and possesses a critical value of the constant $Z = Z_c$ such that a given bound state cannot exist if it is exceeded. The relativistic equation for a hydrogenlike atom is the Dirac equation [7] for a particle with mass μ in the Coulomb field, and it also has a critical constant of the ground state equal to $Z_c = 1/\alpha$. It is reasonable to assume that these constants are the same. This presents the possibility to determine the constant Ω_0 as

$$\Omega_0 = \Omega_c \frac{\hbar^2 c^2}{16(\mu c^2)^4}. \quad (70)$$

In Fig. 2, we plot values of the binding energy for the ground state of a hydrogenlike atom, which is calculated by using the parameter Ω_0 defined in such a way. There, we also present the analogous binding energies according to Schrödinger and Dirac. It is seen that the results of our calculation occupy the intermediate position. Similar calculations can be easily performed for excited levels of hydrogenlike atoms. In this case, levels by Schrödinger with a given n split into n close sublevels since the orbital quantum number l can take n values ($l = 0, 1, \dots, n-1$), i.e., the degeneration is removed in this case. All levels with given n and different l are located under the corresponding Schrödinger level. The value

of the nonrelativistic splitting is much less than that calculated by the Dirac theory. The parameter of noncommutativity for the operators of coordinates and momenta of different particles ε , presented in Fig. 3, increases with the quantum numbers n and l (for the same Z). That is, fully nonrelativistic solutions transfer to that of the Schrödinger equation for large quantum numbers.

As a peculiar feature of a fully nonrelativistic equation, we indicate the presence of a critical value of the parameter αZ_c for any energy level that is not revealed by the Schrödinger nonrelativistic equation. For example, if $n = 2$, $\alpha Z_c = 3$ for $l = 0$ and $\alpha Z_c = 2.5$ for $l = 1$.

When the parameter αZ grows, the mean distance between particles decreases. For the ground state of a hydrogenlike atom, it is defined as

$$\langle |\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1| \rangle = \frac{3\hbar}{2\mu c} \cdot \frac{(1 - \varepsilon_{10})^2}{\alpha Z} \quad (71)$$

and takes the smallest value equal to $\langle |\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1| \rangle \approx 0.9 \frac{\hbar}{\mu c}$ when $\alpha Z = 1$. At the same time (i.e., for $\alpha Z = 1$), mean distances between particles significantly exceed the value $\frac{\hbar}{\mu c}$ for excited quantum states ($\langle |\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1| \rangle \approx 5.9 \frac{\hbar}{\mu c}$ for $n = 2$ and $l = 0$; $\langle |\hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1| \rangle \approx 5 \frac{\hbar}{\mu c}$ for $n = 2$ and $l = 1$). If the parameter αZ further grows, the nonlinear system of Eqs. (62)-(64) has no solutions for the state with $n = 1$, i.e., the 1S state cannot exist, and the ground state is a state with $n = 2$ and $l = 0$ (2S state) or $l = 1$ (2P state). It is possible if $1 < \alpha Z < 3$. Here we are faced with the essential difference from solutions of the Schrödinger nonrelativistic equation, for which, as is well known, the ground state is always the 1S state.

Below, we present the quantum Poisson brackets introduced by Dirac [7] as

$$\{\hat{x}_1, \hat{p}_{1x}\} = 1 - \frac{m_2}{M} \varepsilon, \quad (72)$$

$$\{\hat{x}_2, \hat{p}_{2x}\} = 1 - \frac{m_1}{M} \varepsilon, \quad (73)$$

$$\{\hat{x}_1, \hat{p}_{2x}\} = \frac{m_2}{M} \varepsilon, \quad (74)$$

$$\{\hat{x}_2, \hat{p}_{1x}\} = \frac{m_1}{M} \varepsilon, \quad (75)$$

$$\{\hat{x}_1, \hat{x}_2\} = 0, \quad (76)$$

$$\{\hat{p}_{1x}, \hat{p}_{2x}\} = 0. \quad (77)$$

For $\varepsilon \rightarrow 0$, these brackets transfer to the classical Poisson brackets, i.e., we have the complete analogy between classical and quantum mechanics in this case. As is seen in Fig. 3, ε is remarkably different from zero for systems whose sizes are of the order of the Compton wavelength of particles forming the system. In this case, there is no analogy with classical mechanics. To what extent it would take place can be judged by comparing the proposed theory with experiment. But it is already clear that we have obtained a considerably better agreement with the result of solving the relativistic Dirac equation for the ground state of a hydrogenlike atom as compared with the Schrödinger nonrelativistic theory.

III. A NONRELATIVISTIC SYSTEM OF N -INTERACTING PARTICLES

The previous results can be easily generalized to a system consisting of N particles interacting among themselves via two-particle forces. Let the operators of the coordinates and momenta of N particles be $\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \dots, \hat{\mathbf{r}}_N, \hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \dots, \hat{\mathbf{p}}_N$. Define the operators of coordinates and momentum of the center of masses of the system:

$$\hat{\mathbf{r}}_c = \frac{1}{M} \sum_{k=1}^N m_k \hat{\mathbf{r}}_k, \quad (78)$$

$$\hat{\mathbf{P}}_c = \sum_{k=1}^N \hat{\mathbf{p}}_k. \quad (79)$$

Here, $M = \sum_{k=1}^N m_k$ is the mass of the whole system.

By analogy with a two-particle problem, we require that the commutator of the operator of the coordinate for any particle with the operator of the total momentum of the system be equal to $i\hbar$:

$$[\hat{x}_k, \hat{P}_{cx}] = [\hat{y}_k, \hat{P}_{cy}] = [\hat{z}_k, \hat{P}_{cz}] = i\hbar \quad (k = 1, 2, \dots, N). \quad (80)$$

Then, if

$$[\hat{x}_j, \hat{p}_{kx}] = [\hat{y}_j, \hat{p}_{ky}] = [\hat{z}_j, \hat{p}_{kz}] = i\hbar \frac{m_k}{M} \varepsilon_{jk} \quad (j \neq k), \quad (81)$$

we have

$$[\hat{x}_j, \hat{p}_{jx}] = [\hat{y}_j, \hat{p}_{jy}] = [\hat{z}_j, \hat{p}_{jz}] = i\hbar \left[1 - \sum_{k=1}^N \frac{m_k}{M} \varepsilon_{jk} \right], \quad j = 1, 2, \dots, N. \quad (82)$$

Here, the parameter of noncommutativity of the operators of coordinates and momenta of different particles

$$\varepsilon_{jk} = \frac{\langle \psi, \hat{\varepsilon}(F_{jk}^2, \mu_{jk}) \psi \rangle}{\langle \psi, \psi \rangle} \quad (83)$$

is symmetric with respect to a permutation of the indices j, k and identically equals zero for $j = k$ by definition.

In addition,

$$[\hat{x}_j, \hat{x}_k] = [\hat{y}_j, \hat{y}_k] = [\hat{z}_j, \hat{z}_k] = 0, \quad [\hat{p}_{jx}, \hat{p}_{kx}] = [\hat{p}_{jy}, \hat{p}_{ky}] = [\hat{p}_{jz}, \hat{p}_{kz}] = 0. \quad (84)$$

One of the possible representations for the operators of coordinates and momenta of particles can be written as

$$\hat{\mathbf{r}}_j = \mathbf{r}_j, \quad j = 1, 2, \dots, N, \quad (85)$$

$$\hat{\mathbf{p}}_j = -i\hbar \left[1 - \sum_{q=1}^N \frac{m_q}{M} \varepsilon_{jq} \right] \nabla_j - i\hbar \sum_{k=1}^N \frac{m_j}{M} \varepsilon_{jk} \nabla_k, \quad j = 1, 2, \dots, N. \quad (86)$$

Here, we take coordinates of particles as independent variables since the corresponding operators mutually commute.

In this case, a nonlinear system of equations for the nonrelativistic problem of N particles has the form

$$\left\{ -\frac{\hbar^2}{2} \sum_{i=1}^N \left[\frac{A_i}{m_i} \Delta_i + \sum_{k>i}^N \frac{2B_{ik}}{M} (\nabla_i \cdot \nabla_k) \right] + \sum_{i=1}^N \sum_{j>i}^N V(|\mathbf{r}_j - \mathbf{r}_i|) \right\} \Psi = E\Psi, \quad (87)$$

where

$$A_i = \left(1 - \sum_{q=1}^N \frac{m_q}{M} \varepsilon_{iq} \right)^2 + \sum_{q=1}^N \frac{m_i m_q}{M^2} \varepsilon_{iq}^2, \quad (88)$$

$$B_{ik} = \left(2 - \sum_{q=1}^N \frac{m_q}{M} (\varepsilon_{iq} + \varepsilon_{kq}) \right) \varepsilon_{ik} + \sum_{q=1}^N \frac{m_q}{M} \varepsilon_{iq} \varepsilon_{kq}. \quad (89)$$

Here, ε_{ik} is defined according to (83), and

$$\hat{\varepsilon}(F_{jk}^2, \mu_{jk}) = 1 - \exp \left(-\Omega_c \frac{\hbar^2 c^2}{16(\mu_{jk} c^2)^4} F_{jk}^2 (|\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_k|) \right), \quad (90)$$

$\Omega_c = 0.40765$, and $\mu_{jk} = \frac{m_j m_k}{m_j + m_k}$.

It can be shown that the introduction of the so-called Jacobi coordinates provides the separation of motion of the center of masses as a whole.

The system of equations (87)-(89) takes a particularly simple form in the important case of identical particles ($m_j = m$, $\varepsilon_{jk} = \varepsilon$, $j, k = 1, 2, \dots, N$, $j \neq k$) after the introduction of the so-called normed Jacobi coordinates,

$$\mathbf{q}_k = \sqrt{\frac{k}{k+1}} \left(\frac{1}{k} \sum_{s=1}^k \mathbf{r}_s - \mathbf{r}_{k+1} \right), \quad 1 \leq k \leq N-1, \quad (91)$$

$$\mathbf{q}_N = \sqrt{\frac{1}{N}} \sum_{s=1}^N \mathbf{r}_s. \quad (92)$$

In this case, after the separation of motion of the center of masses, we have

$$\left\{ -\frac{\hbar^2(1-\varepsilon)^2}{2m} (\Delta_{\mathbf{q}_1} + \dots + \Delta_{\mathbf{q}_{N-1}}) + V(\mathbf{q}_1, \dots, \mathbf{q}_{N-1}) \right\} \varphi = E\varphi, \quad (93)$$

$$\varepsilon = \frac{\langle \varphi, \hat{\varepsilon}(F_{12}^2, \frac{m}{2}) \varphi \rangle}{\langle \varphi, \varphi \rangle}. \quad (94)$$

Here, $V(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_{N-1})$ represents the potential energy of the interaction of all particles expressed in terms of Jacobi coordinates (91).

Since $0 \leq \varepsilon < 1$, the mean value of the kinetic energy will be less than that according to Schrödinger, and the energies of the bound states will be situated lower than the Schrödinger ones.

IV. CONCLUSION

The Schrödinger equation for a system of interacting particles is not a strictly nonrelativistic equation because it is grounded on the implicit assumption about finiteness of the interaction propagation velocity. The last means that if the commutator of operators of a coordinate and the corresponding momentum of a free particle is defined as

$$[\hat{x}, \hat{p}_x] = i\hbar, \quad (95)$$

this commutator for a system of coupled particles has the same value $i\hbar$. However, in a nonrelativistic quantum system during measurement of the coordinate of a particle, a whole transferred momentum is distributed over all particles but is not transferred to only the measured one. Therefore, in a system of interacting particles, this commutator should have the form

$$[\hat{x}, \hat{p}_x] = i\hbar\delta, \quad (96)$$

where $0 < \delta \leq 1$.

The rejection of the implicit assumption on finiteness of the propagation velocity of interactions implies the noncommutativity of the operators of coordinates and momenta of different particles. But the operators of coordinates of all particles and operators of momenta of all particles mutually commute that allows one to use these collections as independent variables.

The deduced nonlinear system of integro-differential equations allows one to separate the motion of the center of masses of the system which moves as a free particle.

A solution of this essentially nonlinear system exists for completely definite values of the energy of the system. The wave functions corresponding to these energies, as a rule, are mutually nonorthogonal.

Properties of solutions of the proposed nonlinear system of equations essentially differ from Schrödinger solutions for systems, for which the Compton wavelength of particles is comparable with the size of the system. That is, the consideration of noncommutativity of the operators of coordinates and momenta of different particles is of importance for quantum mechanics of atoms with large charge of the nucleus ($\alpha Z \sim 1$) and for phenomena of the physics of atomic nuclei, where the size of the system is of the order of the Compton wavelength of particles which compose the system.

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FIGURES

FIG. 1. Dependence of the right side of Eq. (69) on η for various values of the parameter $\Omega(\alpha Z)^6$.

FIG. 2. Binding energy of the ground state of a hydrogenlike atom [Eq.(67)] vs the parameter αZ . The upper dotted line corresponds to the Schrödinger theory, $E_S = -\frac{\mu c^2(\alpha Z)^2}{2}$, and the lower one to the Dirac theory, $E_D = \mu c^2 \left\{ -1 + [1 - (\alpha Z)^2]^{1/2} \right\}$.

FIG. 3. Dependence of the parameter of noncommutativity of operators ε on the parameter αZ for the lowest states of a hydrogenlike atom.





